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The Metric Dimension of Algebraic Constructed Graph of Dihedral Group D_n

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Abstract

In this article, we discusses the concept of metric dimension in graph theory and its applications in various scientific fields. Metric dimension is the minimum number of vertices in a graph that can uniquely identify all other vertices based on their distances to the vertices in the set. It has applications in computer science, chemistry, and engineering. We focuses on the study of a specific class of graphs called the algebraic constructed graph of dihedral groups denoted by $\mathcal{D}_n$. The study aims to calculate the resolving sets and metric dimension of these graphs.

Keywords: Resolving set; Metric Dimension; Dihedral Group; Graph Theory; Networks.

1 Introduction

Graph theory is indeed a very versatile and fundamental tool in a wide range of fields, including computer science, image processing, bioinformatics, optimization, game theory, and more. Graphs are composed of vertices and edges, and they can be used to model various real-world situations, such as networks, relationships, and structures. In database design, where

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graphs can be used to represent data dependencies and relationships between tables, that can be helpful to optimize queries and improve database performance. In computer science, graphs are used in shortest path, network flow, and coloring algorithms and also in image processing for tasks such as image segmentation, objects and pattern recognition. In bioinformatics, graphs are used to model biological networks, such as protein-protein interaction networks and gene regulatory networks. Graph algorithms are used to analyze these networks and identify important nodes and relationships. Graphs are also employed in computer networks to study electrical circuits critically and solve challenging puzzles.

Let G be a connected graph, the distance from a vertex u to v is shortest $u-v$ path denoted as $d(u,v)$. A vertex, x is said to resolve two vertices u and v if x has unique distance from each of them, mathematically, $d(u,x) \neq d(v,x)$. Similarly, a subset W of vertex set V is said to resolving set if W has unique distance from each vertex of V. Then the smallest possible resolving set is metric basis and its cardinality is metric dimension of graph.

In graph theory, the concept of locating set was introduced by Slater [12], which was later termed as metric basis by Harary and Melter[5]. The minimum cardinality of a resolving set is known as the locating number/metric basis of a graph. This concept then termed as metric dimension, by Currie and Oellermann[4], an important idea in graph theory with numerous applications in different scientific disciplines such as computer science, chemistry, and engineering. A resolving set is a set of vertices in a graph that uniquely identifies every vertex in the graph based on its distance to the vertices in the set. The metric dimension of a graph is the cardinality of the smallest resolving set of the graph. The problem of finding the metric dimension of a graph is known to be NP-complete[3] i.e to find metric dimension of graph computationally is hard problem. Resolving sets have various applications, including drug discovery, network verification, game strategies, robot navigation, and weighing problems of coins[3, 1, 9]. The concept of resolving sets is also useful in tricky games, processing of maps and images, and station models of SONAR and LORAN[12].

Metric dimension helps in understanding molecular structure, chemical reactivity, bonding pattern as well as drug designs.In drug discovery, metric dimension determines if features of compound are responsible for pharmacological activity. Analyzing the metric dimension of line graph provides insight into arrangements, connectivity of atoms and shed light on funda-

mental properties of graphs [11].

Hussain, Zafar, et al. demonstrated that the 1-pentagonal nanocone network has a constant metric dimension of 3, which has applications in nano-engineering, designing nano-devices from CNC_k, and pharmacy [6]. Dorota Kuziak introduced the concept of strong metric dimension for corona products and join graphs [8]. Imran Shahid, et al. discussed the dimension of rooted product graphs and concluded that some have constant metric dimensions while others have unbounded metric dimensions [7]. Gary Chartrand, et al. determined the metric dimensions for some connected graphs, including the complete graph with a dimension of $n-1$, and P_n with a dimension of 1. They also found that the metric dimension of a tree graph is at least $\delta(G) - ex(G)$, where $\delta(G)$ represents the sum of terminal degree and $ex(G)$ denotes the number of exterior major vertices [3].

In bioinformatics, metric dimension analyze structure of biological molecules e.g DNA sequences as well as proteins. Graphs of protein structures represents amino acids as node and edges as spatial proximity, it identifies key residue that differentiate between protein structures.

In social media, metric dimension identifies the influencers in network and it influences the growth of business. news spread on Facebook,twitter etc affects the marketing campaign. by applying metric dimension we can choose network of minimal set which uniquely identify all other users and help in identifying distinct communities with larger network [13].

The use of algebraic structures in graphical properties is a vast and interesting area of research, with important implications in various fields. For example, in chemistry, chemical structures can be represented as graphs with vertices representing atoms and edges representing bonds. This representation allows for the use of graph theory and algebraic structures to study chemical properties and reactions. In group theory, the study of groups and their properties is important in understanding symmetry and geometric/algebraic structures that can be used to describe the properties of geometric objects and their symmetries.

Pinter came up with an important finite group in algebraic structures in which a regular polygon with n sides forms a group of symmetries, known as dihedral groups, with order twice the number of sides or rotations for any positive integer $n \geq 3$ represented by $\mathcal{D}_n$. The rotation of symmetry for each of the regular n-gons of $\frac{2\pi}{n}$ denoted by as x and the reflection by as y; where the original n-gon is as e then $\mathcal{D}_n = < x,\ y\ | x^n = y^2 = (xy)^2 = e >$. According to Pinter a subgroup $\mathcal{S}$ is itself a group, for any subset $\mathcal{S}$ of $\mathcal{D}_n$ [10].

In this paper we express metric dimension of graphs of subgroups of Dihedral groups. These vertices are adjacent if either of them is contained in other subgroup. As metric dimension depends upon the unique the distances, therefore, here, it is not necessary visuals of graph. Distances can be calculated by examining the containment of subgroups. Maximum distance of each vertex is 2 and minimum distance of vertex is zero(distance of vertex from itself).

2 Dihedral Groups

The Fundamental Theorem of Arithmetic states, every positive integer $n > 1$, can be factor into prime numbers as $n = p_1^{\alpha_1} \times p_2^{\alpha_2} \dots p_k^{\alpha_k}$, then the only positive divisors of n are integers of the form $d = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, where $0 \leq a_i \leq \alpha_i$ for $i = 1, 2, \dots, k$, and that each such divisor generates a subgroup of rotations $\{x^d,\ x^{2d}, \dots, e\}$ in $\mathcal{D}_n$. Therefore, the number of subgroups of $\mathcal{D}_n$ can be determined by finding the number of possible combinations of exponents $a_1, a_2, \dots, a_k$ that satisfy $0 \leq a_i \leq \alpha_i$ for $i = 1, 2, \dots, k$. The number of rotation subgroups is identical to the number of possible such divisors. This allows us to calculate the total number of subgroups of $\mathcal{D}_n$. Consider $\tau(n) = (\alpha_1 + 1)(\alpha_2 + 2) \dots (\alpha_k + 1)$ be number of all divisor and $\sigma(n) = (p_1^{\alpha_1+1} - 1)/(p_1 - 1) \times (p_2^{\alpha_2+1} - 1)/(p_2 - 1) \dots (p_k^{\alpha_k+1} - 1)/(p_k - 1)$ is the sum of divisors of n[2], then $\Delta_n = \sigma(n) + \tau(n)$ represent the number of all subgroups of D(n), see table 1 represent number of all subgroup of $\mathcal{D}_n$ for $n < 40$.

Let $\{H_1,\ H_2, \dots, H_{\Delta_n}\}$ be subgroups of $\mathcal{D}_n$, then we define a graph $\Gamma(\mathcal{D}_n)$ of order Δ_n such that $V(\Gamma(\mathcal{D}_n))$ denoted as vertex set and $E(\Gamma(\mathcal{D}_n))$ as an edge set. Any two vertex $v_i,\ v_j \in V(\Gamma(\mathcal{D}_n))$ are adjacent, i.e. $v_i v_j \in E(\Gamma(\mathcal{D}_n))$, iff H_i is subgroup of H_j vice versa H_j is subgroup of H_i where $i \neq j \in \{1,\ 2, \dots, \Delta_n\}$. The distance of v_i and v_j is length of shortest $v_i v_j$-path denoted by $d(v_i, v_j)$. Let $X = \{v_1,\ v_2, \dots,\ v_s\} \subset V(\Gamma(\mathcal{D}_n))$, then metric representation of any vertex $u \in V(\Gamma(\mathcal{D}_n))$ with respect to subset X imposed ordered s-tuples (s-vectors) denoted by $d(u|X) = (d(u, a_1),\ d(u, a_2), \dots,\ d(u, a_s))$. If $u_1 \neq u_2 \Rightarrow d(u_1|X) \neq r(u_2|X)\ \forall u_1, u_2 \in V(\Gamma(\mathcal{D}_n))$, than the set X is resolving set form the basis for $\Gamma(\mathcal{D}_n)$. Briefly in words,resolving set is subset of vertices that uniquely identifies each vertex based on distances of vertices as well, subset of vertices which has unique distance from each vertex of graph. If the subset X is the metric basis then the cardinality of X is the metric dimension of $\Gamma(\mathcal{D}_n)$, denoted by

$dim(\Gamma(\mathcal{D}_n))$.

Table 1: Number of subgroups of $\mathcal{D}_n$ for $n < 40$.

n	$\boldsymbol{\tau(n)}$	$\boldsymbol{\sigma(n)}$	$\boldsymbol{\Delta(n)}$	**n**	$\boldsymbol{\tau(n)}$	$\boldsymbol{\sigma(n)}$	$\boldsymbol{\Delta(n)}$
3	2	4	6	**21**	4	32	36
4	3	7	10	**22**	4	36	40
6	4	12	16	**24**	8	60	68
7	2	8	10	**25**	3	31	34
8	4	15	19	**26**	4	42	46
9	3	13	16	**27**	4	40	44
10	4	18	22	**28**	6	56	62
11	2	12	14	**29**	2	30	32
12	6	28	34	**30**	8	72	80
13	2	14	16	**31**	2	32	34
14	4	24	28	**32**	6	63	69
15	4	24	28	**33**	4	48	52
16	5	31	36	**34**	4	54	58
17	2	18	20	**35**	4	48	52
18	6	39	45	**36**	9	91	100
19	2	20	22	**37**	2	38	40
20	6	42	48	**38**	4	60	64

3 Methodology

Metric dimension of graph is the minimum number of vertices that uniquely identifies all other vertices as well as we found metric dimension of graphs which were constructed with the help of dihedral groups. Dihedral groups are group of symmetry and number of subgroups of dihedral groups are shown in table. The way to accomplish this is first to define the graph: each vertex is subgroup of Dihedral group, vertices a_1 and a_2 represent trivial subgroups i.e vertex a_1 is dihedral group itself and vertex a_2 is identity subgroup. We took undirected graph to prevent complexity of directed graph. Here it is the graph of dihedral group of order 6 constructed in a way vertex a_1 is dihedral group itself, vertex a_2 is identity subgroup a_3 is group of rotation a_4, a_5, a_6 are groups of reflection.

After labelling the vertices To check whether $i - tuples$ of distance vectors are unique, we construct the following algorithm:

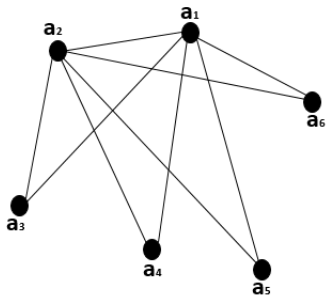

Figure 1: Graph of $\mathcal{D}_3$

1. Distance matrix of the graph is constructed (This table consists of shortest path between pair of vertices).

2. For $1 \leq i \leq n-1, i$ columns are considered at a time. This is done in a total of $\binom{n}{i}$ ways.

3. Among these i vectors if row vectors are unique then metric dimension is i and procedure is finished.

4. Otherwise, choose another $i - column$ and repeat step 3.

5. If all $\binom{n}{i}$ ways are checked then check $i = i + 1$ and repeat from step 2.

Metric dimension is also verified using MATLAB code.

4 Main Results

We proved metric dimension of graphs of subgroups of dihedral groups by taking $n = p, p^2, pq$. Distance table comprises of distances of resolving sets from all vertices of graph and other theorems consists of general formulae for distances.

Theorem 1 *Let $n = p$ be prime, then the cardinality of the resolving set of graph $\Gamma(\mathcal{D}_p)$ is*

$$|X| \geq p + 1.$$

Proof: We know $\mathcal{D}_p$ be dihedral group and H_i be all subgroups of $\mathcal{D}_p$, where $1 \leq i \leq p+3$, then G is graph of subgroups of $\mathcal{D}_(p)$ and each H_i represents the vertices of G. Let $V = \{a_1,\ a_2, \ldots,\ a_{p+3}\}$ for $a_i = H_i$, every a_i connected to a_r iff $H_i \subset H_r$ or $H_r \subset H_i$.

Now consider $X = \{b_1,\ b_{2+j};\quad 1 \leq j \leq p\} \subset V$ of cardinality $p+1$, then we have to show X is resolving set. For this we show any $a_i \in V$ the $(p+1)$-tuples for basis i.e

$$d(a_i|X) = \{d(a_i,\ b_1),\ d(a_i,\ b_3),\ d(a_i,\ b_4), \cdots, d(a_i,\ b_{2+p})\} \tag{1}$$

To show X is resolving set we show the order tuple in Eq.10 give unique identification with all vertices of G, i.e., $d(a_i|X) \neq d(a_k|X),$ for all $i \neq k \in V$.

We formulated general form for distance between resolving set X and set of vertices V, is following

$$d(a_i|X) = \begin{cases} 0 & \text{when } i = j \\ 1 & \text{for } i \text{ or } j = 2 \\ 1 & \text{for any } i \text{ and } j = 1 \\ 2 & \text{otherwise} \end{cases} \tag{2}$$

To prove it we take example of $\mathcal{D}_3$, total subgroups for$\mathcal{D}_3$ are 6 i.e representing in set form as $V = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. As defined earlier X is resolving set consisting of b_1, b_3, b_4, b_5 and adhering to method find tuples of distances

Consider for $i = 1$

$$d(a_1|X) = \{0,\ 1,\ 1,\ 1\}$$

Similarly for $i = 2$,

$$d(a_2|X) = \{1,\ 1,\ 1,\ 1\}$$

Now for $i = 3$,

$$d(a_3|X) = \{1,\ 0,\ 2,\ 2\}$$

Further for $i = 4$

$$d(a_4|X) = \{1,\ 2,\ 0,\ 2\}$$

next for $i = 5$

$$d(a_5|X) = \{1,\ 2,\ 2,\ 0\}$$

proceeding in the same way, lastly consider
$i = \sigma(n) + \tau(n)$ i.e $i = 6$

$$d(a_6|X) = \{1,\ 2,\ 2,\ 2\}$$

Here we see distance of each vertex to resolving set X differs and by excluding any vertex from resolving set for instance, remove b_3 , accordingly we see

$$d(a_3|X) = \{1,\ 2,\ 2\} = d(a_6|X) = \{1,\ 2,\ 2\}$$

Evidently, above expressions manifest that by removing any vertex from resolving set any two vertices will have same distances and not fulfil uniqueness property. Consequently,X is the smallest possible resolving set. □

Theorem 2 *Let G be the graph of subgroups of dihedral group then the metric dimension of graph for $\mathcal{D}_p$ is*

$$|X| = p + 1.$$

Proof: We know $\mathcal{D}_p$ be dihedral group and H_i be all subgroups of $\mathcal{D}_p$, where $1 \leq i \leq p + 3$, then G is graph of subgroups of $\mathcal{D}_p$ and each H_i represents the vertices of G. Let $V = \{a_1,\ a_2, \ldots,\ a_{p+3}\}$ for $a_i = H_i$, every a_i connected to a_r iff $H_i \subset H_r$ or $H_r \subset H_i$.

Now consider $X = \{b_1,\ b_{2+j};\quad 1 \leq j \leq p\} \subset V$ of cardinality $p + 1$, We have to prove that metric dimension is $p + 1$ for $mathcalD_(p)$ where p is prime.

$$d(a_i|X) = \begin{cases} 0 & \text{when } i = j \\ 1 & \text{for } i \text{ or } j = 2 \\ 2 & \text{otherwise} \end{cases} \tag{3}$$

Contrary suppose that metric dimension is p. Then for i=1 either

$$d(a_1|X) = \{0,\ 1,\ 1, \cdots,\ 1\}$$

or

$$d(a_1|X) = \{1,\ 1,\ 1, \cdots,\ 1\}$$

these are p-tuples.

Next for $i = 2$,

$$d(a_2|R) = \{1,\ 1,\ 1, \cdots\ 1\}$$

Now for $i = 3$ either

$$d(a_3|R) = \{1,\ 0,\ 2,\ 2, \cdots,\ 2\}$$

or

$$d(a_3|X) = \{1,\ 2.\ 2, \cdots,\ 2,\}$$

consider for $i = \sigma(n) + \tau(n)$

$$d(a_i|X) = \{1,\ 2,\ 2, \cdots,\ 2\}$$

which shows that there will be any two vertices for which distance is not unique, which concludes that metric dimension is p. □

Theorem 3 *Let G(V,E) = $\mathcal{D}_{p^2}$ is graph of subgroups of dihedral group with metric dimension*

$$|R| \geq p^2 + 1$$

Proof: We know $\mathcal{D}_{p^2}$ be dihedral group and H_i be all subgroups of $\mathcal{D}_{p^2}$, where $1 \leq i \leq \sigma(p^2) + \tau(p^2)$, then G is graph of subgroups of $\mathcal{D}_{p^2}$ and each H_i represents the vertices of G. Let $V = \{a_1,\ a_2, \ldots,\ a_{\sigma(p^2)+\tau(p^2)}\}$ for $a_i = H_i$, every a_i connected to a_r iff $H_i \subset H_r$ or $H_r \subset H_i$.

Now consider $X = \{b_1,\ b_3,\ b_{4+j};\quad 1 \leq j \leq p^2 - 1\} \subset V$ of cardinality p^2+1, then we have to show X is resolving set. For this we show any $a_i \in V$ the $(p^2 + 1)$-tuples for basis i.e

$$d(a_i|X) = \{d(a_i,\ b_1),\ d(a_i,\ b_3),\ d(a_i,\ b_4), \cdots, d(a_i,\ b_{4+p^2-1})\} \tag{4}$$

To show X is resolving set we show the order tuple in Eq.4 give unique identification with all vertice of G, i.e., $d(a_i|X) \neq d(a_k|X)$, for all $i \neq k \in V$. Assume contrary that

$$d(a_i|X) = d(a_k|X) \qquad \text{where} \quad a_i \neq a_k$$

Since we know for any $a_i \in V$ and $b_j \in X$, then

Case 1 $S_1 = \{a_i \quad 1 \leq i \leq \tau(p^2) + \sigma(p^2) - p\}$

$$d(S_1|X) = \begin{cases} 0 & i = j \\ 1 & i, j = 1, i = 2, j = 3 \text{ and } i = 4 \\ 2 & i, j \geq 2 \end{cases} \tag{5}$$

Now consider for $i = 1$

$$d(a_1|X) = \{0,\ 1,\ 1, \cdots,\ 1\}$$

in the same way for $i = 2$

$$d(a_2|X) = \{1,\ 1,\ 1, \cdots,\ 1\}$$

as well for $i = 3$

$$d(a_3|X) = \{1,\ 0,\ 2, \cdots,\ 2\}$$

after that check out for i=4

$$d(a_4|X) = \{1,\ 1,\ 2,\ 2, \cdots,\ 2\}$$

Next for $i = 5$

$$d(a_5|X) = \{1,\ 2,\ 0,\ 2, \cdots,\ 2\}$$

going in the same way examine for $i = \sigma(p^2) + \tau(p^2) - p$

$$d(a_i|X) = \{1,\ 2,\ 2,\ 2, \cdots,\ 2\}$$

We conclude from together above distances that for every single vertex distance is unique, therefore, we can say our supposition is wrong.

$$\Rightarrow d(a_i|X) = d(a_k|X) \text{ only when } i = k$$

Case 2 $S_2 = \{a_i \quad \sigma(P^2) + \tau(p^2) - p < i \leq \sigma(p^2) + \tau(p^2)\}$

$$d(S_2)|R) = \begin{cases} 1 & i \equiv j(mod\ p)\ \forall i \neq j \text{ such that } a_i \in S_3 \\ 2 & i = 3, i \not\equiv j(mod\ p) \end{cases} \tag{6}$$

Studying in the same way, we finalize that there no such vertex exist for which distance is not unique. □

Theorem 4 *Let G be the graph of subgroups of dihedral group then the metric dimension of graph for $\mathcal{D}_{p^2}$ is*

$$P^2 + 1$$

Proof: We know $\mathcal{D}_{p^2}$ be dihedral group and H_i be all subgroups of $\mathcal{D}_{p^2}$, where $1 \leq i \leq \sigma(p^2) + \tau(p^2)$, then G is graph of subgroups of $\mathcal{D}_{p^2}$ and each H_i represents the vertices of G. Let $V = \{a_1,\ a_2, \ldots,\ a_{\sigma(p^2)+\tau(p^2)}\}$ for $a_i = H_i$, every a_i connected to a_r iff $H_i \subset H_r$ or $H_r \subset H_i$.

Now consider $X = \{b_1,\ b_3,\ b_{4+j}; \quad 1 \leq j \leq p^2 - 1\} \subset V$ of cardinality $p^2 + 1$,We have to prove that metric dimension is $p^2 + 1$ for $\mathcal{D}_{p^2}$

$$d(a_i|X) = \{d(a_i,\ b_1),\ d(a_i,\ b_3),\ d(a_i,\ b_4), \cdots, d(a_i,\ b_{4+p^2-1})\} \tag{7}$$

Case 1 $S_1 = \{a_i \quad 1 \leq i \leq \tau(p^2) + \sigma(p^2) - p\}$

$$d(S_1|X) = \begin{cases} 0 & i = j \\ 1 & i, j = 1, i = 2, j = 3 \text{ and } i = 4 \\ 2 & i, j \geq 2 \end{cases} \tag{8}$$

Contrary suppose that metric dimension for $\mathcal{D}_{p^2}$ is p^2. Then consider for $i = 1$

$d(a_1|X) = \{0,\ 1,\ 1, \cdots,\ 1\}$ or $d(a_1|X) = \{1,\ 1,\ 1, \cdots\ 1\}$ p tuples.

now for $i = 2$

$$d(a_2|X) = \{1,\ 1,\ 1, \cdots,\ 1\} \text{ p tuples}$$

likewise for $i = 3$ either

$$d(a_3|X) = \{1,\ 0,\ 2, \cdots,\ 2\}$$

or

$$d(a_3|X) = \{1,\ 2,\ 2, \cdots,\ 2\}$$

or

$$d(a_3|X) = \{0,\ 2,\ 2, \cdots\ 2\} \text{ p tuples}$$

more $i = 4$

$$d(a_4|X) = \{1,\ 2,\ 2, \cdots,\ 2\}$$

or

$$d(a_4|X) = \{1,\ 1,\ 2, \cdots,\ 2\}$$

continuing , at last for $i = \sigma(p^2) + \tau(p^2)_p$

$$d(a_i|X) = \{1,\ 2,\ 2, \cdots,\ 2\}$$

We see there are many possibilities for distance to be not unique so p^2 is not metric dimension. Same will go for case 2.

Case 2 $S_2 = \{a_i \quad \sigma(P^2) + \tau(p^2) - p < i \leq \sigma(p^2) + \tau(p^2)\}$

$$d(S_2)|R) = \begin{cases} 1 & i \equiv j(mod\ p)\ \forall i \neq j \ \text{ such that } a_i \in S_3 \\ 2 & i = 3, i \not\equiv j(mod\ p) \end{cases} \tag{9}$$

□

Theorem 5 *Let G be the graph of subgroups of dihedral group $\mathcal{D}_{pq}$ then there exists a resolving set X of G such that*

$$|X| \geq p + q + 1,$$

where p and q are prime.

Proof: We know $\mathcal{D}_{pq}$ be dihedral group and H_i be all subgroups of $\mathcal{D}_{pq}$, where

$$1 \leq i \leq \sigma(pq) + \tau(pq),$$

then G is graph of subgroups of $\mathcal{D}_{pq}$ and each H_i represents the vertices of G. Let $V = \{a_1,\ a_2, \ldots,\ a_{\sigma(pq)+\tau(pq}\}$ for $a_i = H_i$, every a_i connected to a_r iff $H_i \subset H_r$ or $H_r \subset H_i$.

Now consider $X = \{b_1,\ b_3,\ b_j; \quad pq + 7 \leq j \leq \sigma(pq) + \tau(pq)\} \subset V$ of cardinality $p + q + 1$, then we have to show X is resolving set. For this we show any $a_i \in V$ the $(p + q + 1)$-tuples for basis i.e

$$d(a_i|X) = \{d(a_i,\ b_1),\ d(a_i,\ b_3),\ d(a_i,\ b_4), \cdots, d(a_i,\ b_j)\} \tag{10}$$

To show X is resolving set we show the order tuple in Eq.10 gives unique identification with all vertices of G, i.e., $d(a_i|X) \neq d(a_k|X),$ for all $i \neq k \in V$.

Table 2: Distance of vertices of $\mathcal{D}_6$ to resolving set

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
a_1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a_3	1	1	0	1	1	2	2	2	2	2	2	2	2	2	2	2
a_{13}	1	1	2	2	1	2	1	2	2	1	2	2	0	2	2	2
a_{14}	1	1	2	2	1	2	2	1	2	2	1	2	2	0	2	2

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
a_{15}	1	1	2	1	2	1	2	1	2	1	2	2	2	2	0	2
a_{16}	1	1	2	1	2	2	1	2	1	2	1	2	2	2	2	0

Table 3: Distance of vertices of $\mathcal{D}_{10}$ to resolving set

	a_1	a_3	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}	a_{22}
a_1	o	1	1	1	1	1	1	1
a_2	1	1	1	1	1	1	1	1
a_3	1	0	1	1	1	1	1	1
a_4	1	1	2	2	2	2	1	1
a_5	1	1	1	1	1	1	2	2
a_6	1	2	2	2	2	2	1	2
a_7	1	2	1	2	2	2	2	1
a_8	1	2	2	1	2	2	1	2
a_9	1	2	2	2	1	2	2	1
a_{10}	1	2	2	2	2	1	1	2
a_{11}	1	2	1	2	2	2	2	1
a_{12}	1	2	1	2	2	2	1	2
a_{13}	1	2	2	1	2	2	2	1
a_{14}	1	2	2	2	1	2	1	2
a_{15}	1	2	2	2	2	1	2	1
a_{16}	1	2	2	2	2	2	2	2
a_{17}	1	2	0	2	2	2	2	2
a_{18}	1	2	2	0	2	2	2	2
a_{19}	1	2	2	2	0	2	2	2
a_{20}	1	2	2	2	2	0	2	2
a_{21}	1	2	2	2	2	2	0	2
a_{22}	1	2	2	2	2	2	2	0

Table 4: Distance of vertices of $\mathcal{D}_{21}$ and next $n = pq$ to resolving set

	a_1	a_3	a_{28}	a_{29}	a_{30}	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}
a_1	0	1	1	1	1	1	1	1	1	1	1
a_2	1	1	1	1	1	1	1	1	1	1	1

	a_1	a_3	a_{28}	a_{29}	a_{30}	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}
a_3	1	0	2	2	2	2	2	2	2	2	2
a_4	1	1	2	2	2	2	2	2	1	1	1
a_5	1	1	1	1	1	1	1	1	2	2	2
a_6	1	2	2	2	2	2	2	2	1	2	2
a_7	1	2	1	2	2	2	2	2	2	1	2
a_8	1	2	2	1	2	2	2	2	2	2	1
a_9	1	2	2	2	1	2	2	2	1	2	2
a_{10}	1	2	2	2	2	1	2	2	2	1	2
a_{11}	1	2	2	2	2	2	1	2	2	2	1
a_{12}	1	2	2	2	2	2	2	1	1	2	2
a_{13}	1	2	2	2	2	2	2	2	2	1	2
a_{14}	1	2	1	2	2	2	2	2	2	2	1
a_{15}	1	2	2	1	2	2	2	2	1	2	2
a_{16}	1	2	2	2	1	2	2	2	2	1	2
a_{17}	1	2	2	2	2	1	2	2	2	2	1
a_{18}	1	2	2	2	2	2	1	2	1	2	2
a_{19}	1	2	2	2	2	2	2	1	2	1	2
a_{20}	1	2	2	2	2	2	2	2	2	2	1
a_{21}	1	2	1	2	2	2	2	2	1	2	2
a_{22}	1	2	2	1	2	2	2	2	2	1	2
a_{23}	1	2	2	2	1	2	2	2	2	1	1
a_{24}	1	2	2	2	2	1	2	2	1	2	2
a_{25}	1	2	2	2	2	2	1	2	2	1	2
a_{26}	1	2	2	2	2	2	2	1	2	2	1
a_{27}	1	2	2	2	2	2	2	2	2	2	2
a_{28}	1	2	0	2	2	2	2	2	2	2	2
a_{29}	1	2	2	0	2	2	2	2	2	2	2
a_{30}	1	2	2	2	0	2	2	2	2	2	2
a_{31}	1	2	2	2	2	0	2	2	2	2	2
a_{32}	1	2	2	2	2	2	0	2	2	2	2
a_{33}	1	2	2	2	2	2	2	0	2	2	2
a_{34}	1	2	2	2	2	2	2	2	0	2	2
a_{35}	1	2	2	2	2	2	2	2	2	0	2
a_{36}	1	2	2	2	2	2	2	2	2	2	0

From above tables we conclude that none of the vertices have same distance.

□

Theorem 6 *Let G be the graph of subgroups of dihedral group $\mathcal{D}_{pq}$ then the metric dimension of G is*

$$p + q + 1,$$

where p and q are prime.

Proof: We know $\mathcal{D}_{pq}$ be dihedral group and H_i be all subgroups of $\mathcal{D}_{pq}$, where

$$1 \leq i \leq \sigma(pq) + \tau(pq),$$

then G is graph of subgroups of $mathcalD_{pq}$ and each H_i represents the vertices of G. Let $V = \{a_1,\ a_2, \ldots,\ a_{\sigma(pq)+\tau(pq)}\}$ for $a_i = H_i$, every a_i connected to a_r iff $H_i \subset H_r$ or $H_r \subset H_i$.

Now consider $X = \{b_1,\ b_3,\ b_j; \quad pq + 7 \leq j \leq \sigma(pq) + \tau(pq)\} \subset V$ of cardinality $p + q + 1$, we have to prove that metric dimension is $p + q + 1$ for $\mathcal{D}_{pq}$.

Table 5: Distance of vertices of $\mathcal{D}_6$ to resolving set

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
a_1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a_3	1	1	0	1	1	2	2	2	2	2	2	2	2	2	2	2
a_{13}	1	1	2	2	1	2	1	2	2	1	2	2	0	2	2	2
a_{14}	1	1	2	2	1	2	2	1	2	2	1	2	2	0	2	2
a_{15}	1	1	2	1	2	1	2	1	2	1	2	2	2	2	0	2
a_{16}	1	1	2	1	2	2	1	2	1	2	1	2	2	2	2	0

Table 6: Distance of vertices of $\mathcal{D}_{10}$ to resolving set

	a_1	a_3	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}	a_{22}
a_1	o	1	1	1	1	1	1	1
a_2	1	1	1	1	1	1	1	1
a_3	1	0	1	1	1	1	1	1
a_4	1	1	2	2	2	2	1	1
a_5	1	1	1	1	1	1	2	2
a_6	1	2	2	2	2	2	1	2
a_7	1	2	1	2	2	2	2	1

	a_1	a_3	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}	a_{22}
a_8	1	2	2	1	2	2	1	2
a_9	1	2	2	2	1	2	2	1
a_{10}	1	2	2	2	2	1	1	2
a_{11}	1	2	1	2	2	2	2	1
a_{12}	1	2	1	2	2	2	1	2
a_{13}	1	2	2	1	2	2	2	1
a_{14}	1	2	2	2	1	2	1	2
a_{15}	1	2	2	2	2	1	2	1
a_{16}	1	2	2	2	2	2	2	2
a_{17}	1	2	0	2	2	2	2	2
a_{18}	1	2	2	0	2	2	2	2
a_{19}	1	2	2	2	0	2	2	2
a_{20}	1	2	2	2	2	0	2	2
a_{21}	1	2	2	2	2	2	0	2
a_{22}	1	2	2	2	2	2	2	0

Table 7: Distance of vertices of $\mathcal{D}_{21}$ and next $n = pq$ to resolving set

	a_1	a_3	a_{28}	a_{29}	a_{30}	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}
a_1	0	1	1	1	1	1	1	1	1	1	1
a_2	1	1	1	1	1	1	1	1	1	1	1
a_3	1	0	2	2	2	2	2	2	2	2	2
a_4	1	1	2	2	2	2	2	2	1	1	1
a_5	1	1	1	1	1	1	1	1	2	2	2
a_6	1	2	2	2	2	2	2	2	1	2	2
a_7	1	2	1	2	2	2	2	2	2	1	2
a_8	1	2	2	1	2	2	2	2	2	2	1
a_9	1	2	2	2	1	2	2	2	1	2	2
a_{10}	1	2	2	2	2	1	2	2	2	1	2
a_{11}	1	2	2	2	2	2	1	2	2	2	1
a_{12}	1	2	2	2	2	2	2	1	1	2	2
a_{13}	1	2	2	2	2	2	2	2	2	1	2
a_{14}	1	2	1	2	2	2	2	2	2	2	1
a_{15}	1	2	2	1	2	2	2	2	1	2	2

	a_1	a_3	a_{28}	a_{29}	a_{30}	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}
a_{16}	1	2	2	2	1	2	2	2	2	1	2
a_{17}	1	2	2	2	2	1	2	2	2	2	1
a_{18}	1	2	2	2	2	2	1	2	1	2	2
a_{19}	1	2	2	2	2	2	2	1	2	1	2
a_{20}	1	2	2	2	2	2	2	2	2	2	1
a_{21}	1	2	1	2	2	2	2	2	1	2	2
a_{22}	1	2	2	1	2	2	2	2	2	1	2
a_{23}	1	2	2	2	1	2	2	2	2	1	1
a_{24}	1	2	2	2	2	1	2	2	1	2	2
a_{25}	1	2	2	2	2	2	1	2	2	1	2
a_{26}	1	2	2	2	2	2	2	1	2	2	1
a_{27}	1	2	2	2	2	2	2	2	2	2	2
a_{28}	1	2	0	2	2	2	2	2	2	2	2
a_{29}	1	2	2	0	2	2	2	2	2	2	2
a_{30}	1	2	2	2	0	2	2	2	2	2	2
a_{31}	1	2	2	2	2	0	2	2	2	2	2
a_{32}	1	2	2	2	2	2	0	2	2	2	2
a_{33}	1	2	2	2	2	2	2	0	2	2	2
a_{34}	1	2	2	2	2	2	2	2	0	2	2
a_{35}	1	2	2	2	2	2	2	2	2	0	2
a_{36}	1	2	2	2	2	2	2	2	2	2	0

From the above tables it is clear that if any one of the vertex is removed from resolving set then remaining set of vertices is not metric dimension. □

5 Conclusion

Overall, the article provides mathematical proofs and results regarding the metric dimension of algebraic constructed graphs of $\mathcal{D}_n$. The findings can be useful in various scientific disciplines, particularly in optimization, network analysis, and drug discovery. The study contributes to the understanding of metric dimension and its applications in different fields. The article also discusses the graph of subgroups of $\mathcal{D}_{pq}$ and shows the existence of a resolving set for this graph. The metric dimension of the graph is then proven to be $p + q + 1$.

6 Future Work

The reader can explore the finding for other values of n that can be p^2qr, p^2q etc.

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